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Space-Time Symmetries of Superstring and Jordan Algebras

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The sequence of Jordan algebras $\mathfrak{M}_3^n$, whose elements are the 3×3 Hermitian matrices over the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{O}$, is considered. These algebras are naturally related to supersymmetric structures in space-time dimensions of 3, 4, 6, and 10, as the Lorentz groups in these dimensions can be expressed in a unified way as a subgroup of the structure group of the Jordan algebras $\mathfrak{M}_3^n$. The generators of the complete structure group and the automorphism group can be separated into bosonic and fermionic generators, depending on their transformation properties under the Lorentz subgroup. A peculiar connection between these fermionic generators and the supersymmetry generators of the superstring action is introduced and discussed.

1. INTRODUCTION

The first quantized superstring (Green and Schwarz, 1984) or superparticle (Brink and Schwarz, 1981; Casalbuoni, 1976) action has a number of characteristic space-time symmetries. These include global super-Poincaré invariance, and also a local fermionic and a local bosonic symmetry. The space-time symmetries of the action are indeed plentiful. Classically, the superstring action can be defined in dimensions 3, 4, 6, and 10. The space-time Lorentz groups in these dimensions can be expressed in a unified way in terms of the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{O}$, and, consequently, the formulation of the Lorentz group in terms of the division algebras allows for a unified description of several sequences of physical theory (Kugo and Townsend, 1983; Sudbery, 1984; Hasiewicz and Lukierski, 1984; Chung and Sudbery, 1987; Gursey, 1987; Foot and Joshi, 1988*a*,*b*; Bengtsson and Cederwall, 1988; Kimura and Oda, 1988; Oda *et al.*, 1988; Manogue and Sudbery, 1988). One way of expressing these four Lorentz groups in terms of the four division algebras is by extracting these Lorentz groups as subgroups of the structure groups of the sequence of four Jordan algebras

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$\mathfrak{M}_3^1$, $\mathfrak{M}_3^2$, $\mathfrak{M}_3^4$, $\mathfrak{M}_3^8$ (Foot and Joshi, 1987). The classical superstring theories in light cone gauge were expressed in terms of these Jordan algebras in Foot and Joshi (1987). A motivation for reexpressing these theories in terms of the Jordan algebras or (equivalently) the division algebras is that these formulations only exist in the classically allowed dimensions of these theories, and thus it seems that this framework may make it easier to develop a deeper understanding of these theories. It is also intriguing that the exceptional Jordan algebra $\mathfrak{M}_3^8$ corresponds to the $d = 10$ case, which is the critical dimension of the superstring. The connection between the exceptional algebra $\mathfrak{M}_3^8$ and the $D = 10$ superstring has recently received new emphasis in relation to the algebra of vertex operators (Goddard *et al.*, 1987; Corrigan and Hollowood, 1988*a*,*b*; Ferreira *et al.*, 1988; Gunaydin and Hyun, 1988).

The outline of this paper is as follows. Section 2 is devoted to establishing our notation and reviewing the formulation of the Lorentz groups in terms of the subgroup of the structure groups of the Jordan algebras. In Section 3, we consider the covariant superstring action and reformulate that action in terms of the Jordan algebra. In Section 4, we discuss the decomposition of the automorphism group and the structure group in terms of the Lorentz subgroup. In Section 5, a toy string Lagrangian is considered, whereby the symmetries of the toy Lagrangian are related to the structure group transformations. In Section 6, it is shown that the toy model construction of Section 5 can be extended in a rather peculiar way to the case of the superstring action in light-cone gauge. Finally, in Section 7 we conclude.

2. JORDAN ALGEBRA AND THE LIGHT-CONE SUPERSTRING

Jordan algebra was initially introduced by Jordan (1933) and further developed by Jordan *et al.* (1934). These authors were concerned with early problems of the quantum theory, which have long since been accounted for. Contemporary efforts to utilize Jordan algebra have focused on the exceptional Jordan algebra $\mathfrak{M}_3^8$ (Gunaydin and Gursey, 1973; Gursey, 1975; Beidenharn and Truini, 1981; Nambu, 1973) [see also the review in Sorgsepp and Lohmus, 1979).

The formally real Jordan algebra can be defined axiomatically as follows:

$$A \cdot B = B \cdot A \qquad \text{(commutativity)} \tag{1a}$$

$$(A, B, A^2) = 0 \qquad \text{(Jordan identity)} \tag{1b}$$

$$A^2 + B^2 = 0 \;\Rightarrow\; A \text{ or } B = 0 \qquad \text{(reality)} \tag{1c}$$

where we have introduced the associator defined by

$$(A, B, C) = (A \cdot B) \cdot C - A \cdot (B \cdot C) \tag{2}$$

Observe that the algebra is commutative, but not associative. However, the algebra is power associative, i.e., $a^n A^m = A^{n+m}$ (which is a consequence of the above axioms). Every finite-dimensional Jordan algebra, with one exception, can be expressed in terms of real matrices. The product is the Jordan product:

$$A \circ B = \tfrac{1}{2}(AB + BA) \tag{3}$$

where the product on the right-hand side is the usual matrix multiplication. The exception is the algebra $\mathfrak{M}_3^8$, whose elements are the 3×3 Hermitian matrices over the octonions (with the Jordan product).

There are several ways in which Lie groups can be defined by the Jordan algebras (Koecher, 1967; Gunaydin, 1979). The automorphism group of a Jordan algebra consists of the group of linear transformations preserving its multiplication table. These transformations can be expressed as follows:

$$M' = M + \frac{1}{1!}(A, M, B) + \frac{1}{2!}(A, (A, M, B), B) + \cdots \tag{4}$$

where A, B are traceless elements of the particular Jordan algebra under consideration. For the exceptional Jordan algebra $\mathfrak{M}_3^8$, the automorphism group is the Lie group F_4. The Lie algebra defined by the automorphism group is called the derivation algebra of the Jordan algebra. A generalization of the automorphism group is the reduced structure group, with infinitesimal transformations of the form

$$\delta M = (A, M, B) + C \circ M \tag{5a}$$

$$\delta \bar{M} = (A, \bar{M}, B) - C \circ \bar{M} \tag{5b}$$

where A, B, C are traceless elements of the Jordan algebra. The Lie algebra defined by the reduced structure group will be denoted as the structure algebra of the Jordan algebra. The representations M and $\bar{M}$ transform contragrediently with respect to the form (i.e., this form is invariant)

$$I = \mathrm{Tr}\, M \circ \bar{M} \tag{6}$$

Finite transformations can be obtained in the usual way by exponentiation of the infinitesimal transformations

$$\delta_{\text{finite}} M = \delta M + \frac{1}{2!}\delta(\delta M) + \cdots \tag{7}$$

For the sequence of Jordan algebras $\mathfrak{M}_3^1$, $\mathfrak{M}_3^2$, $\mathfrak{M}_3^4$, and $\mathfrak{M}_3^8$, the automorphism and reduced structure groups are given in Table I. The Lie groups in this table form part of the Freudenthal–Tits magic square (Schafer, 1966).

Table I. The Automorphism and Reduced Structure Groups of the Jordan Algebras $\mathfrak{M}_3^n$

Group	$\mathfrak{M}_3^1$	$\mathfrak{M}_3^2$	$\mathfrak{M}_3^4$	$\mathfrak{M}_3^8$
Automorphism group	$SO(3)$	$SU(3)$	$Sp(6)$	F_4
Reduced structure group	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$

For the most part, we will consider the generic Jordan algebra $\mathfrak{M}_3^n$, where $\mathfrak{M}_3^n$ can be either $\mathfrak{M}_3^1$, $\mathfrak{M}_3^2$, $\mathfrak{M}_3^4$, or $\mathfrak{M}_3^8$. A basis for $\mathfrak{M}_3^n$ can be expressed as

$$A_f=\begin{pmatrix}0 & e_f & 0\\ e_f^* & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad B_f=\begin{pmatrix}0 & 0 & e_f\\ 0 & 0 & 0\\ e_f^* & 0 & 0\end{pmatrix},\qquad C_f=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & e_f\\ 0 & e_f^* & 0\end{pmatrix}$$
$$\mathbb{1}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\qquad J_1=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad J_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} \tag{8}$$

where $f=0,\ldots,d(\mathbb{A})-1$.

The covering group of the Lorentz group in $D=3$, 4, 6, and 10 dimensions can be expressed as a subgroup of the reduced structure group of the Jordan algebras $\mathfrak{M}_3^n$. The infinitesimal action of the Lorentz group on an element of the algebra $\mathfrak{M}_3^n$ is (Foot and Joshi, 1987; Gamba, 1967)

$$\delta M=\xi^{fk}(A_f, M, A_k)+\xi_1^k(J_1, M, A_k)+\xi_2^k A_k\circ M+\xi_3 J_1\circ M \tag{9a}$$

$$\delta\bar{M}=\xi^{fk}(A_f, \bar{M}, A_k)+\xi_1^k(J_1, \bar{M}, A_k)-\xi_2^k A_k\circ\bar{M}-\xi_3 J_1\circ\bar{M} \tag{9b}$$

and $f, k=0,\ldots,d(\mathbb{A})-1$. The representation M breaks up, under the action of the subgroup (9), into the irreducible representations

$$X=\begin{pmatrix}x_{11} & x_{12} & 0\\ x_{12}^* & x_{22} & 0\\ 0 & 0 & 0\end{pmatrix},\qquad \Lambda=\begin{pmatrix}0 & 0 & \lambda_1\\ 0 & 0 & \lambda_2\\ \lambda_1^* & \lambda_2^* & 0\end{pmatrix},\qquad \Phi=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \phi\end{pmatrix} \tag{10}$$

The representations X, Λ, and Φ correspond to the vector, spinor, and scalar of the Lorentz group. Similar results hold for the representation $\bar{M}$, where the irreducible representations under the subgroup generated by the transformation (9b) can be denoted as in (10), but with the indices raised, i.e.,

$$\bar{X}=\begin{pmatrix}x^{11} & x^{12} & 0\\ x^{12*} & x^{22} & 0\\ 0 & 0 & 0\end{pmatrix},\qquad \bar{\Lambda}=\begin{pmatrix}0 & 0 & \lambda^{1*}\\ 0 & 0 & \lambda^{2*}\\ \lambda^1 & \lambda^2 & 0\end{pmatrix},\qquad \bar{\Phi}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \phi'\end{pmatrix} \tag{11}$$

Dirac matrices can be introduced by first defining generalized Pauli matrices:

$$\sigma^0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \sigma^{d(\mathbb{A})+1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\sigma^{i+1}=\begin{pmatrix}0 & e_i\\ e_i^* & 0\end{pmatrix}, \qquad i=0,\ldots,d(\mathbb{A})-1$$

$$\bar\sigma^0=\sigma^0, \qquad \bar s^j=-\sigma^j, \qquad j=1,\ldots,d(\mathbb{A})+1 \tag{12}$$

The σ^μ matrices have $SL(2,\mathbb{A})$ index structure:

$$\sigma^\mu \leftrightarrow \sigma^{\mu\alpha\dot\alpha}, \qquad \bar\sigma^\mu \leftrightarrow \bar\sigma^{\mu\dot\alpha\alpha} \tag{13}$$

For $\mathbb{A}=\mathbb{C}$ these matrices correspond to the usual Pauli matrices. In the Jordan formalism these matrices have to be embedded into the elements of the Jordan algebra $\mathfrak{M}_3^n$ as follows:

$$\Sigma^\mu=\begin{pmatrix}\Sigma^\mu & & 0\\ & & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \bar\Sigma^\mu=\begin{pmatrix}\bar\Sigma^\mu & & 0\\ & & 0\\ 0 & 0 & 0\end{pmatrix} \tag{14}$$

The Σ^μ matrices transform like vectors, which is analogous to the $SL(2,\mathbb{C})$ formalism. Observe that X defined in (10) can be expressed as $X=X_\mu\Sigma^\mu$, which is analogous to the $SL(2,\mathbb{C})$ formalism. Denote by $\Lambda^{(n)}$ the spinor associated with $\mathfrak{M}_3^n$. Then there exist the following correspondences:

$$\begin{aligned}&\Lambda^{(1)}\sim d=3 \text{ Majorana spinor}\\ &\Lambda^{(2)}\sim d=4 \text{ Weyl spinor}\\ &\Lambda^{(4)}\sim d=6 \text{ Weyl spinor}\\ &\Lambda^{(8)}\sim d=10 \text{ Majorana-Weyl spinor}\end{aligned} \tag{15}$$

In order to complete the translation of the usual $S\tilde{O}(n+1,1)$ formalism to that of $\mathfrak{M}_3^n$, we will list the correspondence of various Lorentz invariants. We will take the spinors to be anticommuting, i.e., the coefficients of each element of the algebra $\mathbb{A}$ will be Grassmann. The Jordan product only operates on the Jordan matrix part, i.e., if J_1, J_2 are two elements of the Jordan algebra, then $J_1\circ J_2=J_1^m J_2^n A_m\circ A_n$ (here the A_m form a basis of the Jordan algebra, i.e., $J_1=J_1^m A_m$, $J_2=J_2^m A_m$). If we consider two spinors Λ_1 and Λ_2 with components θ, λ, respectively, then

$$\begin{aligned}\mathrm{Tr}(\bar\Lambda_1\circ\Lambda_2)&=2\,\mathrm{Re}(\theta^\alpha\lambda_\alpha)\sim i(\bar\psi_1\psi_2+\bar\psi_2\psi_1)\\ \mathrm{Tr}(\Lambda_1\circ\bar\Sigma^\mu\circ\Lambda_2)&=\mathrm{Re}(\theta_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\lambda_\beta)\sim\tfrac12(\bar\psi_1\gamma^\mu\psi_2-\bar\psi_2\gamma^\mu\psi_1)\\ \mathrm{Tr}(\bar\Lambda_1\circ\Sigma^\mu\circ(\bar\Sigma^\nu\circ\Lambda_2))&=\tfrac12\,\mathrm{Re}(\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\sigma^{\nu\dot\beta\gamma}\lambda_\gamma)\\ &\sim\frac{i}{4}(\bar\psi_1\gamma^\mu\gamma^\nu\psi_2+\bar\psi_2\gamma^\nu\gamma^\mu\psi_1)\end{aligned} \tag{16}$$

where Re denotes the real part and Tr denotes the trace. The invariants of $SL(2, \mathbb{A})$ thus appear as the component form of the equations of the Jordan formulation of the Lorentz group. The last correspondence indicates the relation with the usual $2^{D/2}$ component formalism, where ψ is Majorana-Weyl for $D=10$, Weyl for $D=6$, Majorana or Weyl for $D=4$ [if ψ is taken to be Majorana, then a further factor of 1/2 should be included on the right-hand side of equation (16) for this case], and Majorana for $D=3$. We note in passing that there is an identity for Majorana spinors which will simplify this last correspondence:

$$\bar{\psi}_1\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\psi_2=(-1)^n\bar{\psi}_2\gamma^{\mu_n}\gamma^{\mu_{n-1}}\cdots\gamma^{\mu_1}\psi_1 \tag{17}$$

To apply the $\mathfrak{M}_3^n$ formalism to the superstring in light-cone gauge (Green and Schwarz, 1982), we require the transverse (light-cone preserving) subgroup of the Lorentz group. The infinitesimal transformations are given by the first term equation of (9):

$$\delta M=\varepsilon^{fk}(A_f, M, A_k) \tag{18}$$

Here, $M=X_T$, Λ_1, Λ_2, where

$$X_T=X^iA_i, \qquad \Lambda_1=\Lambda_1^iB_i, \qquad \Lambda_2=\Lambda_2^iC_i \tag{19}$$

and the matrices A_i, B_i, and C_i are defined in (8). The light-cone conditions are

$$X^{11}=x^{11}+2\alpha' p^{11}\tau, \qquad \Lambda_2=0 \tag{20}$$

Note that $X^{11}=2^{1/2}X^+$. The action for the light-cone (classical) superstring is

$$S=\mathrm{Tr}\int d\sigma\, d\tau\, \frac{1}{8\pi}\,\partial_\alpha X_T\circ\partial^\alpha X_T+\frac{i}{4\pi}\,\tilde{\Lambda}_1\circ\bar{\Sigma}^-\circ\rho\cdot\partial\Lambda_1 \tag{21}$$

Here Λ_1 is a world-sheet spinor (the world-sheet indices are suppressed) as well as a space-time spinor, $\tilde{\Lambda}_1=\Lambda_1\rho^0$, and $\bar{\Sigma}^-=(1/\sqrt{2})\ (\bar{\Sigma}^0-\bar{\Sigma}^9)$. In Foot and Joshi (1987) we proceeded to find the equations of motion and introduced the spectrum of states. Quantization was defined through quantization of the coefficients-Λ_1^i and X^j defined in (19). As is well known, quantization of equation (21) is only consistent for $d=10$ (Goddard *et al.*, 1973), which intriguingly corresponds to the exceptional Jordan algebra $\mathfrak{M}_3^8$.

3. THE COVARIANT SUPERSTRING ACTION

Hitherto, we have analyzed the superstring in light-cone gauge. We will now examine the covariant superstring action (Green and Schwarz, 1984). This action has global Poincaré invariance, as well as local fermionic and a local bosonic symmetry. We here reformulate this action in terms of the Jordan algebraic formalism. The action is

$$S=\int d\sigma\, d\tau\,(L_1+L_2) \tag{22}$$

where

$$L_1=\tfrac{1}{4}(\sqrt{-g}\,g^{\alpha\beta}\,\mathrm{Tr}\,\bar{\Pi}_\alpha\circ\Pi_\beta) \tag{23a}$$

$$\begin{aligned}L_2=&-i\varepsilon^{\alpha\beta}\{\mathrm{Tr}(\Lambda^1\circ\partial_\alpha\bar{X}\circ\partial_\beta\Lambda^1)-\mathrm{Tr}(\Lambda^2\circ\partial_\alpha\bar{X}\circ\partial_\beta\Lambda^2)\}\\&+\varepsilon^{\alpha\beta}\,\mathrm{Tr}(\Lambda^1\circ\bar{\Sigma}^\mu\circ\partial_\alpha\Lambda^1)\,\mathrm{Tr}(\Lambda^2\circ\bar{\Sigma}_\mu\circ\partial_\beta\Lambda^2)\end{aligned} \tag{23b}$$

and

$$\Pi_\alpha=\partial_\alpha X-i\Sigma_\mu\,\mathrm{Tr}(\Lambda^A\circ\bar{\Sigma}^\mu\circ\partial_\alpha\Lambda^A) \tag{24}$$

Here $A=1,2$ is an internal supersymmetry index, and X and Λ are defined in (10). The action has the following symmetries: a global space-time supersymmetry defined by

$$\begin{aligned}\delta\Lambda^A&=\xi^A\\ \delta X&=i\Sigma_\mu\,\mathrm{Tr}(\xi^A\circ\bar{\Sigma}^\mu\circ\Lambda^A)\end{aligned} \tag{25}$$

and a local fermionic symmetry [or Siegel (1983) symmetry] given by

$$\begin{aligned}\delta_K\Lambda^1&=4i\Pi_\alpha P_-^{\alpha\beta}\circ\bar{K}_\beta\\ \delta_K\Lambda^2&=4i\Pi_\alpha P_+^{\alpha\beta}\circ\bar{K}_\beta\\ \delta_K X&=i\Sigma_\mu\,\mathrm{Tr}\,\Lambda^A\circ\bar{\Sigma}^\mu\circ\delta_K\Lambda^A\\ \delta_K(\sqrt{-g}\,g^{\alpha\beta})&=8\sqrt{-g}\,(P_-^{\alpha\gamma}P_-^{\beta\delta}\,\mathrm{Tr}\,\bar{K}_\delta^1\circ\partial_\gamma\Lambda^1+P_+^{\alpha\gamma}P_+^{\beta\delta}\,\mathrm{Tr}\,\bar{K}_\delta^2\circ\partial_\gamma\Lambda^2)\end{aligned} \tag{26}$$

where

$$P_\pm^{\alpha\beta}=\tfrac{1}{2}(g^{\alpha\beta}\pm\varepsilon^{\alpha\beta}/\sqrt{-g}) \tag{27}$$

The action is also invariant under the local bosonic transformations

$$\begin{aligned}\delta_\lambda\Lambda^1&=\sqrt{-g}\,P_-^{\alpha\beta}\partial_\beta\Lambda^1\lambda_\alpha\\ \delta_\lambda\Lambda^2&=\sqrt{-g}\,P_+^{\alpha\beta}\partial_\beta\Lambda^2\lambda_\alpha\\ \delta_\lambda X&=i\Sigma_\mu\,\mathrm{Tr}\,\Lambda^A\circ\bar{\Sigma}^\mu\circ\delta_\lambda\Lambda^A\end{aligned} \tag{28}$$

Invariance of L_1 under the global supersymmetry transformations (25) is straightforward. Invariance of L_2 is more complex, and requires the identity

$$\begin{aligned}&\bar{\Sigma}_\mu\circ\Lambda_1\,\mathrm{Tr}(\Lambda_2\circ\bar{\Sigma}^\mu\circ\Lambda_3)+\bar{\Sigma}_\mu\circ\Lambda_2\,\mathrm{Tr}(\Lambda_3\circ\bar{\Sigma}^\mu\circ\Lambda_1)\\&\quad+\bar{\Sigma}_\mu\circ\Lambda_3\,\mathrm{Tr}(\Lambda_1\circ\bar{\Sigma}^\mu\circ\Lambda_2)=0\end{aligned} \tag{29}$$

This is analogous to the identity

$$\gamma_\mu \psi_1 \bar{\psi}_2 \gamma^\mu \psi_3 + \gamma_\mu \psi_2 \bar{\psi}_3 \gamma^\mu \psi_1 + \gamma_\mu \psi_3 \bar{\psi}_1 \gamma^\mu \psi^2 = 0 \tag{30}$$

which arises in the usual spinor formalism in the proof of invariance of L_2 and also in the proof of supersymmetry for super-Yang-Mills theories (Brink *et al.*, 1977). The usual proof of equation (30) requires the use of Fierz transformations, and holds only for $d=3$ Majorana spinors, $d=4$ Majorana spinors, $d=6$ Weyl spinors, and $d=10$ Majorana-Weyl spinors. The proof of equation (29) in this Jordan framework is very simple, and can be shown to hold from simple properties of the division algebras (Fairlie and Manogue, 1987). Invariance of the action under the other symmetries (26) and (28) is reasonably straightforward to check. In doing so, we again require the identity (29), and also the identities

$$(\bar{\Sigma}^\mu, \Sigma^\nu, \bar{K}) + (\bar{\Sigma}^\nu, \Sigma^\mu, \bar{K}) = -\tfrac{1}{2}\eta^{\mu\nu}\bar{K} \tag{31a}$$

$$\bar{\Sigma}^\mu \circ \Sigma^\nu + \bar{\Sigma}^\nu \circ \Sigma^\mu = -2\eta^{\mu\nu} L \tag{31b}$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

and $\eta^{\mu\nu}$ has signature $(-+++\cdots)$.

4. DECOMPOSITION OF THE STRUCTURE GROUP

The infinitesimal transformations of the automorphism and structure groups can be separated into two types. We will focus on the structure group; the corresponding results for the automorphism group can be easily obtained, as the automorphism group is a subgroup of the structure group. The two types of transformations can be classified as follows:

$$\text{L type:} \quad \delta(X+\Phi) = \begin{pmatrix} \alpha & a & 0 \\ a^* & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \delta\Lambda = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ b^* & c^* & 0 \end{pmatrix} \tag{33}$$

$$\text{S type:} \quad \delta(X+\Phi) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ b^* & c^* & 0 \end{pmatrix}, \quad \delta\Lambda = \begin{pmatrix} \alpha & a & 0 \\ a^* & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \tag{34}$$

Recall that X, Φ, and Λ are defined in (10). The L-type transformations include the Lorentz transformations given in the previous sections. The L- and S-type transformations are given below.

L Type
Lorentz:

$$\delta M = \xi^{fk}(A_f, M, A_k) + \xi_1^k(J_1, M, A_k) + \xi_2^k A_k \circ M + \xi_3 J_1 \circ M$$

Phase:

For $d=4$, $\mathfrak{M}_3^2$: $\delta M = \frac{4}{3}\eta(B_0, M, B_1) + \frac{4}{3}\eta(C_0, M, C_1)$

For $d=6$, $\mathfrak{M}_3^4$: $\delta M = \eta^i(B_0, M, B_i) - \frac{1}{2}\eta^l \varepsilon_{lij}(B_i, M, B_j)$

$U(1)$:

$$\delta M = J_2 \circ M \tag{35}$$

S Type

S_1: $\delta M = 2\varepsilon_1^k(J_1, M, B_k) - 2\varepsilon_2^k(J_1, M, C_k)$

$$S_2: \delta M = K_1^k B_k \circ M + K_2^k C_k \circ M \tag{36}$$

Recall that A_k, B_k, C_k, J_1, and J_2 are defined in (8). Here i, j, l run from 1 to 3. In component form these phase transformations correspond to $\delta x = 0$, $\delta\lambda_\alpha = i\eta\lambda_\alpha$ for $\mathfrak{M}_3^2$, and the quaterionic counterpart $\delta x = 0$, $\delta\lambda_\alpha = -\eta^i e_i \lambda_\alpha$ $(i = 1, \ldots, 3)$. The algebras $\mathfrak{M}_3^1$ and $\mathfrak{M}_3^8$ do not have analogous phase rotations. The algebra $\mathfrak{M}_3^1$ obviously does not, as its elements are real matrices. The algebra $\mathfrak{M}_3^8$ might be expected to have an associated octonionic phase. However, such a phase cannot exist as an automorphism, which is a consequence of the nonassociativity of the octonions. Note that for $L \in \{$L type$\}$ and $S \in \{$S type$\}$, then the following Lie brackets hold:

$$[L, L] \in L \tag{37a}$$

$$[L, S] \in S \tag{37b}$$

$$[S, S] \in L \tag{37c}$$

The transformations associated with the structure group or the automorphism group of the Jordan algebras can be interpreted physically, as we will show in the following sections.

5. A TOY MODEL

Consider the following toy string action:

$$S = \mathrm{Tr}\,\frac{1}{8\pi}\int d\sigma\, d\tau\, \partial_\alpha \bar{X} \circ \partial^\alpha X + \partial_\alpha \bar{\Lambda} \circ \partial^\alpha \Lambda + \partial_\alpha \bar{\Phi} \circ \partial^\alpha \Phi \tag{38}$$

where X, Λ, Φ are defined in (10). The variables in this toy Lagrangian are the usual position coordinate X, which is a space-time vector, but a world-sheet scalar, while Λ is a space-time commuting spinor and a world-sheet scalar and Φ is a space-time and world-sheet scalar. The action (38) is invariant under the $SO(9, 1)$ Lorentz transformations (9a), (9b). There are also phase rotations for the $d=4$ and $d=6$ actions given by

$$\text{for } d=4: \qquad \delta M = \tfrac{4}{3}\eta(B_0, M, B_1) + \tfrac{4}{3}\eta(C_0, M, C_1) \tag{39a}$$

$$\text{for } d=6: \qquad \delta M = \eta^i(B_0, M, B_i) - \tfrac{1}{2}\eta^l \varepsilon_{lij}(B_i, M, B_j) \tag{39b}$$

where $M = X$, Λ, Φ. The action is also invariant under the following fermionic transformations:

$$\delta X = \Sigma^\mu \operatorname{Tr}(\xi \circ \bar{\Sigma}_\mu \circ \Lambda), \qquad \delta \bar{X} = 0 \tag{40a}$$

$$\delta \Phi = 0, \qquad \delta \bar{\Phi} = -P \operatorname{Tr}(\bar{\Lambda} \circ \xi) \tag{40b}$$

$$\delta \Lambda = \xi \operatorname{Tr} \Phi, \qquad \delta \bar{\Lambda} = -2\bar{X} \circ \xi \tag{40c}$$

and

$$\delta X = 0, \qquad \delta \bar{X} = -\bar{\Sigma}^\mu \operatorname{Tr}(\bar{\Lambda} \circ \Sigma_\mu \circ \bar{\eta}) \tag{41a}$$

$$\delta \Phi = P \operatorname{Tr}(\bar{\eta} \circ \Lambda), \qquad \delta \bar{\Phi} = 0 \tag{41b}$$

$$\delta \Lambda = 2X \circ \bar{\eta}, \qquad \delta \bar{\Lambda} = -\bar{\eta} \operatorname{Tr} \Phi \tag{41c}$$

where P is the projection

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{42}$$

Observe that these fermionic transformations treat X, Φ, Λ and $\bar{X}$, $\bar{\Phi}$, $\bar{\Lambda}$ as independent variables. The conserved currents associated with the above symmetries can be calculated via Noether's procedure. The currents are

$$\begin{aligned} J^\alpha = -\frac{1}{8\pi} \operatorname{Tr}(&\partial^\alpha \bar{X} \circ \delta X + \partial^\alpha \bar{\Lambda} \circ \delta \Lambda + \partial^\alpha \bar{\Phi} \circ \delta \Phi \\ &+ \delta \bar{X} \circ \partial^\alpha X + \delta \bar{\Lambda} \circ \partial^\alpha \Lambda + \delta \bar{\Phi} \circ \partial^\alpha \Phi) \end{aligned} \tag{43}$$

and the corresponding charges are given by

$$\begin{aligned} \int_0^\pi J^0 \, d\sigma = -\frac{1}{8\pi} \int_0^\pi &\operatorname{Tr}(\partial^0 \bar{X} \circ \delta X + \partial^0 \bar{\Lambda} \circ \delta \Lambda + \partial^0 \bar{\Phi} \circ \delta \Phi \\ &+ \delta \bar{X} \circ \partial^0 X + \delta \bar{\Lambda} \circ \partial^0 \Lambda + \delta \bar{\Phi} \circ \partial^0 \Phi) \, d\sigma \end{aligned} \tag{44}$$

where δX, $\delta \Lambda$, $\delta \Phi$ and $\delta \bar{X}$, $\delta \bar{\Phi}$, $\delta \bar{\Lambda}$ are given by (9) for the Lorentz current and charge, (39) for the phase, and (40) and (41) for the currents and charges associated with the fermionic symmetries.

The algebra of these generators is given by the structure algebra. The symmetry associated with the infinitesimal transformations defined by the structure algebra can be realized manifestly, by combining X, Λ, and Φ to form the field $\mathfrak{G} = \mathrm{X} + \Lambda + \Phi$. In terms of $\mathfrak{G}$,

$$\mathfrak{L} = \frac{1}{8\pi} \partial_\alpha \bar{\mathfrak{G}} \, \partial^\alpha \mathfrak{G} \tag{45}$$

and the Lorentz, phase, and fermionic transformations are given by

$$\delta \mathfrak{G} = (A, \mathfrak{G}, B) + C \circ \mathfrak{G} \tag{46a}$$

$$\delta \bar{\mathfrak{G}} = (A, \bar{\mathfrak{G}}, B) - C \circ \bar{\mathfrak{G}} \tag{46b}$$

The fermionic transformations $\bar{\eta}$ and ξ are given in terms of the S-type transformations (36) by

$$\xi + \bar{\eta} \equiv \varepsilon_1^k B_k + \varepsilon_2^k C_k \tag{47a}$$

$$\bar{\eta} - \xi \equiv K_1^k B_k + K_2^k C_k \tag{47b}$$

6. A GENERALIZATION OF THE TOY MODEL TO THE SUPERSTRING ACTION

An alternative description of the Lagrangian in Section 5 which we will generalize to the case of the superstring can be defined as follows:

$$\begin{aligned} \mathfrak{L} = \frac{1}{8\pi} \mathrm{Tr}\{&\partial_\alpha \bar{X} \circ \partial^\alpha X + \partial_\alpha \bar{\Lambda} \circ \partial^\alpha \Lambda + \partial_\alpha \bar{\Phi} \circ \partial_\alpha \Phi \\ &+ (\partial_\alpha \bar{X} + \partial_\alpha \bar{\Phi}) \circ \partial^\alpha \Lambda + \mathrm{Tr}(\partial_\alpha X + \partial_\alpha \Phi) \circ \partial^\alpha \bar{\Lambda}\} \end{aligned} \tag{48}$$

Equation (48) arises if equation (45) is expanded out and all the terms are included, even those which would usually be neglected (as they have zero trace), such as the last two terms in equation (48). Thus, instead of defining the one field $\mathfrak{G}$, we consider the three fields X, Λ, Φ. The transformations generated by

$$\delta X = (A, X, B) + C \circ X \tag{49a}$$

$$\delta \Phi = (A, \Phi, B) + C \circ \Phi \tag{49b}$$

$$\delta \Lambda = (A, \Lambda, B) + C \circ \Lambda \tag{49c}$$

reproduce the Lorentz, phase, and fermionic currents, even though the S-type transformations of (49) cannot be expressed in component form. It is this peculiar description of the symmetries of the toy model which we can generalize to other, more interesting theories.

Consider first the superstring action in light-cone gauge. The supersymmetry algebra for this action is of course the super-Poincaré algebra. Since the action in light-cone gauge has only manifest transverse Lorentz invariance, we consider the derivation algebra rather than the structure algebra. This Lie algebra has nothing *a priori* to do with the symmetries of this superstring action. Nevertheless, a peculiar connection can be made, by generalizing this description of the toy model. We consider the following superstring action:

$$\mathfrak{L}_T = \mathfrak{L} + \mathfrak{L}' \tag{50}$$

where $\mathfrak{L}$ is given in (21) and $\mathfrak{L}'$ is given by

$$\mathfrak{L}' = \frac{1}{\pi} \operatorname{Tr} \partial_\alpha \tilde{\Lambda}_1 \circ \rho^\alpha \rho \cdot \partial \bar{X} \theta \tag{51}$$

where θ is an arbitrary (constant) $d=2$ (world-sheet) Majorana spinor and space-time scalar. This term analogues the last two terms of equation (48). Thus, while $\mathfrak{L}'$ is manifestly invariant under the transformations defined by the derivation Lie algebra associated with the sequence of Jordan algebras $\mathfrak{M}_3^n$, the fermionic transformations are nontrivially invariant in the sense that these transformations produce a nonvanishing current. This current is precisely the global supersymmetry current of the superstring in light-cone gauge. Furthermore, the addition of the term $\mathfrak{L}'$ does not disturb the bosonic currents. The fermionic current derived from $\mathfrak{L}'$ (and hence $\mathfrak{L}+\mathfrak{L}'$) is

$$J_s^\alpha = \frac{1}{\pi} \operatorname{Tr} \tilde{\Lambda}_1 \circ \rho^\alpha \rho \cdot \partial \bar{X} \circ \xi \tag{52}$$

where $\xi = \varepsilon\theta$ is a world-sheet spinor and space-time spinor. Note that the spinor ξ can be assumed to be anticommuting without changing the essence of our arguments. In deriving equation (52), care must be taken in applying Noether's procedure. This is because the fermionic transformations do not have the form of a variation, which is a consequence of the result that the transformations cannot be expressed in component form. (This result should be compared with the toy model, where the transformations could be expressed in component form by introducing the field $\mathfrak{G}$.) It turns out that the current is conserved if and only if the following identity is satisfied:

$$\operatorname{Tr}\{(\partial\mathfrak{L}/\partial M) \circ \delta M\} - \operatorname{Tr}\{\partial_\alpha[\partial\mathfrak{L}/\partial(\partial_\alpha M)] \circ \delta M\} = 0 \tag{53}$$

where $M = X$, Λ_1, or Φ. For ordinary transformations, equation (53) vanishes by virtue of the equations of motion. In the case at hand, (53) is indeed satisfied when the equations of motion hold. It is obvious that this must be the result, because $\mathfrak{L}'$ was constructed so that the current of equation (52) should be equivalent to the usual supersymmetry current. Actually, the extra term $\mathfrak{L}'$ is uniquely specified modulo surface terms (which lead to the same charge when the appropriate boundary conditions are taken) by demanding that the current be nontrivial, conserved when the equations of motion hold [i.e., equation (53) is satisfied], and that the term be only bilinear in the fields X, Λ. We expect from physical considerations that his last condition should be unnecessary, but we have not proved this.

In this section we have only considered the superstring in the light-cone gauge, and have shown how the symmetries can be related to the Lie algebras associated with the automorphism groups as explained in Section 4. We now comment on the covariant superstring action, which was expressed in the Jordan algebra framework in Section 3 by extracting the Lorentz groups in 3, 4, 6, and 10 dimensions as subgroups of the structure groups of the Jordan algebras $\mathfrak{M}_3^n$. It thus appears that the Lie algebra defined by the structure group is the appropriate starting point. This algebra contains two fermionic generators (which we denote by ξ and $\bar{\eta}$) which transform contragrediently. The result that the structure algebra contains two fermionic generators and that they transform contragrediently to each other is intriguing because two such fermionic symmetries arise in the covariant superstring action. However, the local K transformation of the covariant superstring action is rather peculiar, as this transformation cannot be derived from a conserved current (Green *et al.*, 1987). It is therefore unclear whether there exists any correspondence between the fermionic symmetries of the covariant superstring action and the structure algebras of the Jordan algebras $\mathfrak{M}_3^n$.

7. CONCLUDING REMARKS

The structure groups of the sequence of four Jordan algebras $\mathfrak{M}_3^n$ contain the Lorentz groups in 3, 4, 6, and 10 dimensions. These dimensions correspond to the four classical superstring theories. A consequence of formulating the classical superstrings in terms of the Jordan algebraic framework is that the four classical superstring theories appear in one-to-one correspondence to the four division algebras. In this formulation one is naturally interested to know whether the complete structure group or the complete automorphism group may be related to all of the symmetries of the superstring. We have found a way of making such a connection, although the usefulness and implications of this connection remain unclear.

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